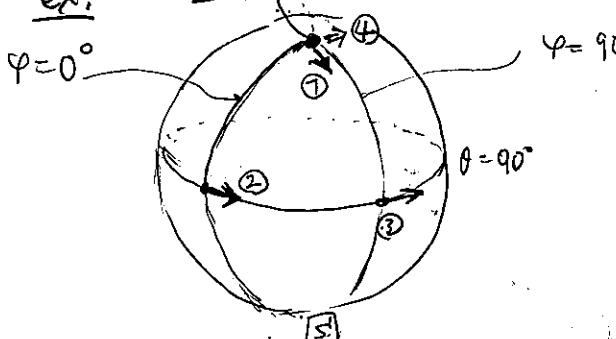


§5 曲率

曲面上の開曲線を三回して任意のベクトルを平行移動

ex.

圖 $\theta=0^\circ$



一般に移動前後のベクトルは重なる

2次元球面上で

$$\textcircled{1} \quad \theta = \varphi = 0^\circ \text{ で } \varphi = 90^\circ \text{ 方向へ向くベクトル}$$



$$\textcircled{2} \quad \varphi = 0^\circ \text{ で } \theta = 90^\circ \text{ で } \theta = 90^\circ \pm 2^\circ \text{ 平行移動}$$



$$\textcircled{3} \quad \theta = 90^\circ \text{ で } \varphi = 90^\circ \pm 2^\circ$$



$$\textcircled{4} \quad \varphi = 90^\circ \text{ で } \theta = 0^\circ \pm 2^\circ$$

逆に、開曲線を3回して

(平行移動したベクトル)

が元のベクトルに重なる

性質を使ひ、空間の曲がり方"を定義

⇒ 曲率テンソル

5-1 Riemann テンソル

- 平行移動 \Rightarrow 接続 (共変微分)
- 開曲線3回の移動 \Rightarrow 微分の交換

ベクトルの異なった方向への共変微分と交換を表す。

* スカラーフィールドは可換

(4)

$$\begin{aligned} \nabla_\mu \psi &= \partial_\mu \psi \rightarrow \nabla_\nu \nabla_\mu \psi = \nabla_\nu (\partial_\mu \psi) = \partial_\nu (\partial_\mu \psi) - \Gamma_{\mu\nu}^\lambda \partial_\lambda \psi \\ &= \partial_\mu (\partial_\nu \psi) - \Gamma_{\nu\mu}^\lambda \partial_\lambda \psi \\ &= \nabla_\mu (\nabla_\nu \psi) \end{aligned}$$

$$\begin{aligned}
 \nabla_\alpha \nabla_\beta V^\mu &= \nabla_\alpha (\nabla_\beta V^\mu) = \partial_\alpha (\nabla_\beta V^\mu) - \Gamma_{\beta\alpha}^\lambda (\nabla_\lambda V^\mu) + \Gamma_{\lambda\alpha}^\mu (\nabla_\beta V^\lambda) \\
 &= \partial_\alpha (\partial_\beta V^\mu + \Gamma_{\nu\beta}^\mu V^\nu) - \Gamma_{\beta\alpha}^\lambda (\partial_\lambda V^\mu + \Gamma_{\nu\lambda}^\mu V^\nu) + \Gamma_{\lambda\alpha}^\mu (\partial_\beta V^\lambda + \Gamma_{\nu\beta}^\lambda V^\nu) \\
 &= \underbrace{\partial_\alpha \partial_\beta V^\mu}_{\alpha \leftrightarrow \beta} + \underbrace{\partial_\alpha \Gamma_{\nu\beta}^\mu V^\nu}_{\alpha \leftrightarrow \beta} + \underbrace{\Gamma_{\nu\beta}^\mu \partial_\alpha V^\nu}_{\alpha \leftrightarrow \beta} - \underbrace{\Gamma_{\beta\alpha}^\lambda \partial_\lambda V^\mu}_{\alpha \leftrightarrow \beta} - \underbrace{\Gamma_{\beta\alpha}^\lambda \Gamma_{\nu\lambda}^\mu V^\nu}_{\alpha \leftrightarrow \beta} + \underbrace{\Gamma_{\lambda\alpha}^\mu \partial_\beta V^\lambda}_{\alpha \leftrightarrow \beta} + \underbrace{\Gamma_{\lambda\alpha}^\mu \Gamma_{\nu\beta}^\lambda V^\nu}_{\alpha \leftrightarrow \beta}
 \end{aligned}$$

$$\begin{aligned}
 (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu &= (\underbrace{\partial_\alpha \Gamma_{\nu\beta}^\mu - \partial_\beta \Gamma_{\nu\alpha}^\mu}_{R^M_{\nu\alpha\beta}} + \Gamma_{\lambda\alpha}^\mu \Gamma_{\nu\beta}^\lambda - \Gamma_{\lambda\beta}^\mu \Gamma_{\nu\alpha}^\lambda) V^\nu \\
 &\equiv R^M_{\nu\alpha\beta} \cdot V^\nu \quad \textcircled{1}
 \end{aligned}$$

①の左辺は AA だから $(\frac{1}{2})$ テンソルなので、右辺も $(\frac{1}{2})$ テンソル

V^ν は任意なので $R^M_{\nu\alpha\beta}$ は $(\frac{1}{3})$ テンソルの成分。

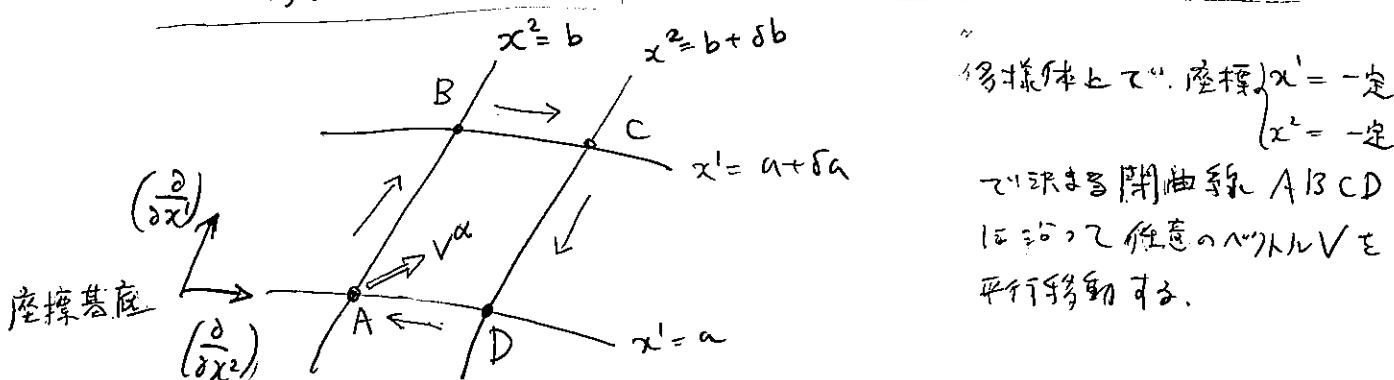
$$R^M_{\nu\alpha\beta} = \partial_\alpha \Gamma_{\nu\beta}^\mu - \partial_\beta \Gamma_{\nu\alpha}^\mu + \Gamma_{\lambda\alpha}^\mu \Gamma_{\nu\beta}^\lambda - \Gamma_{\lambda\beta}^\mu \Gamma_{\nu\alpha}^\lambda \quad \textcircled{2}$$

曲率テンソル (Riemann テンソル)

5-2 Riemann テンソルとベクトルの平行移動

観点を変えて、ベクトルを開曲線で平行移動した

場合を考える。



$V_{\parallel}^{\alpha}(A) \in ABCD$ に沿って平行移動した結果のベクトルを $V_{\parallel}^{\alpha}(A)$ とする。 (3)

$$V_{\parallel}^{\alpha}(A) - V^{\alpha}(A) = \delta V^{\alpha}(A) \quad (\text{同一の点 } x^1 \text{ のベクトルの差分})$$

この δV^{α} の表式を求める。

$A \rightarrow B, C \rightarrow D$ の経路 x^1 は $(\frac{\partial}{\partial x^1})$ に沿って平行移動 ($x^2 = \text{一定}$)

$$\nabla_{(\frac{\partial}{\partial x^1})} V^{\alpha} = 0 \quad \longrightarrow \quad \frac{\partial V^{\alpha}}{\partial x^1} + \Gamma_{\mu 1}^{\alpha} V^{\mu} = 0$$

$$(\text{基底ベクトル } (\frac{\partial}{\partial x^1}) \text{ に沿って } V^{\alpha} \text{ が平行移動}) \quad \Downarrow \quad \frac{\partial V^{\alpha}}{\partial x^1} = - \Gamma_{\mu 1}^{\alpha} V^{\mu} \quad (3)$$

(3) で $A \rightarrow B$ に対する積分

$$V^{\alpha}(B) = V^{\alpha}(A) - \int_A^B \Gamma_{\mu 1}^{\alpha} V^{\mu} dx^1 \Big|_{x^2=b} \quad (4)$$

同様に、

$$B \rightarrow C \quad V^{\alpha}(C) = V^{\alpha}(B) - \int_B^C \Gamma_{\mu 2}^{\alpha} V^{\mu} dx^2 \Big|_{x^1=a+\delta a} \quad (5)$$

$$C \rightarrow D \quad V^{\alpha}(D) = V^{\alpha}(C) - \int_C^D \Gamma_{\mu 1}^{\alpha} V^{\mu} dx^1 \Big|_{x^2=b+\delta b}$$

$$= V^{\alpha}(D) + \int_D^C \Gamma_{\mu 2}^{\alpha} V^{\mu} dx^2 \Big|_{x^1=a} \quad (6)$$

$$D \rightarrow A \quad V_{\parallel}^{\alpha}(A) = V^{\alpha}(D) + \int_A^D \Gamma_{\mu 2}^{\alpha} V^{\mu} dx^2 \Big|_{x^1=a} \quad (7)$$

④

④) — ② も辺々足し.

$$\delta V^\alpha(A) = \int_A^D \Gamma_{\mu 2}^\alpha V^\mu dx^2 \Big|_{x^1=a} + \int_D^C \Gamma_{\mu 1}^\alpha V^\mu dx^1 \Big|_{x^2=b+\delta b}$$

$$- \int_B^C \Gamma_{\mu 2}^\alpha V^\mu dx^2 \Big|_{x^1=a+\delta a} - \int_A^B \Gamma_{\mu 1}^\alpha V^\mu dx^1 \Big|_{x^2=b} \quad (8)$$

 $\approx 2\pi^4$

$$- \int_B^C \Gamma_{\mu 2}^\alpha V^\mu dx^2 \Big|_{x^1=a+\delta a} = - \left[\int_A^D \Gamma_{\mu 2}^\alpha V^\mu dx^2 \right]_{x^1=a} + \delta a \cdot \frac{\partial}{\partial x^1} \left\{ \int_A^D \Gamma_{\mu 2}^\alpha V^\mu dx^2 \right\} \Big|_{x^1=a} + O((\delta a)^2)$$

 $x^1=a$ の Taylor 展開

$$\int_D^C \Gamma_{\mu 1}^\alpha V^\mu dx^1 \Big|_{x^2=b+\delta b} = \int_A^B \Gamma_{\mu 1}^\alpha V^\mu dx^1 \Big|_{x^2=b} + \delta b \cdot \frac{\partial}{\partial x^2} \left\{ \int_A^B \Gamma_{\mu 1}^\alpha V^\mu dx^1 \right\} \Big|_{x^2=b} + O((\delta b)^2)$$

$$\therefore \delta V^\alpha(A) = -\delta a \cdot \frac{\partial}{\partial x^1} \left\{ \int_A^D \Gamma_{\mu 2}^\alpha V^\mu dx^2 \right\} \Big|_{x^1=a} + \delta b \cdot \frac{\partial}{\partial x^2} \left\{ \int_A^B \Gamma_{\mu 1}^\alpha V^\mu dx^1 \right\} \Big|_{x^2=b} + O((\delta a)^2) + O((\delta b)^2)$$

更に

$$\frac{\partial}{\partial x^1} \int_A^D \Gamma_{\mu 2}^\alpha V^\mu dx^2 = \int_A^D \frac{\partial}{\partial x^1} (\Gamma_{\mu 2}^\alpha V^\mu) dx^2 = \frac{\partial}{\partial x^1} (\Gamma_{\mu 2}^\alpha V^\mu) \cdot \delta b + O((\delta b)^2)$$

左辺から

$$(9) \Rightarrow \delta V^\alpha = \delta a \cdot \delta b \cdot \left\{ -\frac{\partial}{\partial x^1} (\Gamma_{\mu 2}^\alpha V^\mu) + \frac{\partial}{\partial x^2} (\Gamma_{\mu 1}^\alpha V^\mu) \right\} + O(3) \quad (10)$$

(10) の $\frac{\partial}{\partial x^1} (\)$ を計算して.

$$\frac{\partial}{\partial x^1} V^\mu = -\Gamma_{\beta 1}^\mu V^\beta \quad (\text{平行移動!})$$

 $(\delta a)(\delta b)^2$ etc.左辺を計算して整理する。更に、座標 $\Gamma_1 = \Gamma_k$, $\Gamma_2 = \Gamma_\lambda$ と置く。

$$\delta V^\alpha = \delta a \cdot \delta b \cdot V^\beta \cdot \left\{ \frac{\partial}{\partial x^1} \Gamma_{\beta k}^\alpha - \frac{\partial}{\partial x^k} \Gamma_{\beta \lambda}^\alpha + \Gamma_{\mu \lambda}^\alpha \Gamma_{\beta k}^\mu - \Gamma_{\mu k}^\alpha \Gamma_{\beta \lambda}^\mu \right\}$$

$$R_{\beta \lambda k}^{\alpha} \quad //$$

(11)

5-3 Riemann テンソルの対称性.

(i) 表式② より.

$$\underline{\underline{R^{\mu}_{\nu\beta\alpha}}} = -\underline{\underline{R^{\mu}_{\nu\alpha\beta}}} \quad (12)$$

(ii), ② より

$$R^{\mu}_{\nu\alpha\beta} = \cancel{\partial_{\alpha}\Gamma^{\mu}_{\nu\beta}}^1 - \cancel{\partial_{\beta}\Gamma^{\mu}_{\nu\alpha}}^2 + \cancel{\Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\nu\beta}}^4 - \cancel{\Gamma^{\mu}_{\lambda\beta}\Gamma^{\lambda}_{\nu\alpha}}^5$$

$$R^{\mu}_{\alpha\beta\nu} = \cancel{\partial_{\beta}\Gamma^{\mu}_{\alpha\nu}}^2 - \cancel{\partial_{\nu}\Gamma^{\mu}_{\alpha\beta}}^3 + \cancel{\Gamma^{\mu}_{\beta\lambda}\Gamma^{\lambda}_{\alpha\nu}}^5 - \cancel{\Gamma^{\mu}_{\lambda\nu}\Gamma^{\lambda}_{\alpha\beta}}^6$$

$$R^{\mu}_{\beta\nu\alpha} = \cancel{\partial_{\nu}\Gamma^{\mu}_{\beta\alpha}}^3 - \cancel{\partial_{\alpha}\Gamma^{\mu}_{\beta\nu}}^1 + \cancel{\Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\beta\alpha}}^6 - \cancel{\Gamma^{\mu}_{\lambda\alpha}\Gamma^{\lambda}_{\beta\nu}}^4$$

(+)

$$\underline{\underline{R^{\mu}_{\nu\alpha\beta}}} + \underline{\underline{R^{\mu}_{\alpha\beta\nu}}} + \underline{\underline{R^{\mu}_{\beta\nu\alpha}}} = 0 \quad (13)$$

サインルール 運用

(iii) ② より 添字 μ を下に記す

$$R_{\gamma\nu\alpha\beta} = g_{\gamma\mu} R^{\mu}_{\nu\alpha\beta} = g_{\gamma\mu} \left\{ \partial_{\alpha}\Gamma^{\mu}_{\nu\beta} + \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\nu\beta} \right\} - [\alpha \leftrightarrow \beta] \quad (14)$$

222nd 次の記述を定義 (Christoffel 組)

前2項 $\gamma \leftrightarrow \beta$
を交換した

$$\Gamma_{\gamma\alpha\beta} \equiv g_{\gamma\mu} \Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} (\partial_{\beta}g_{\alpha\gamma} + \partial_{\alpha}g_{\gamma\beta} - \partial_{\gamma}g_{\alpha\beta}) \quad (15)$$

$$\begin{aligned} (14) : R_{\gamma\nu\alpha\beta} &= \partial_{\alpha}\Gamma_{\gamma\nu\beta} - \partial_{\beta}\Gamma_{\gamma\nu\alpha} + \Gamma_{\gamma\lambda\alpha}\Gamma^{\lambda}_{\nu\beta} - [\alpha \leftrightarrow \beta] \\ &= \partial_{\alpha}\Gamma_{\gamma\nu\beta} - \partial_{\alpha}g_{\gamma\lambda}\Gamma^{\lambda}_{\nu\beta} + \Gamma_{\gamma\lambda\alpha}\Gamma^{\lambda}_{\nu\beta} - [\alpha \leftrightarrow \beta] \end{aligned}$$

$$= \partial_{\alpha}\Gamma_{\gamma\nu\beta} + (\Gamma_{\gamma\lambda\alpha} - \partial_{\alpha}g_{\gamma\lambda})\cdot\Gamma^{\lambda}_{\nu\beta} - [\alpha \leftrightarrow \beta] \quad (16)$$

$$\begin{aligned} (16) \text{ で } \Gamma_{\gamma\lambda\alpha} - \partial_{\alpha}g_{\gamma\lambda} &= \frac{1}{2} (\partial_{\alpha}g_{\gamma\lambda} + \partial_{\lambda}g_{\gamma\alpha} - \partial_{\gamma}g_{\lambda\alpha}) - \partial_{\alpha}g_{\gamma\lambda} \\ &= \frac{1}{2} (\partial_{\lambda}g_{\gamma\alpha} - \partial_{\alpha}g_{\gamma\lambda} - \partial_{\gamma}g_{\lambda\alpha}) \\ &= -\Gamma_{\lambda\gamma\alpha} \end{aligned}$$

(16)

$$\begin{aligned}
 R_{\gamma\nu\alpha\beta} &= \partial_\alpha \Gamma_{\gamma\nu\beta} - \Gamma_{\lambda\gamma\alpha} \Gamma_{\nu\beta}^\lambda - [\alpha \leftrightarrow \beta] \\
 &= \partial_\alpha \Gamma_{\gamma\nu\beta} - \partial_\beta \Gamma_{\gamma\nu\alpha} - \Gamma_{\lambda\gamma\alpha} \Gamma_{\nu\beta}^\lambda + \Gamma_{\lambda\gamma\beta} \Gamma_{\nu\alpha}^\lambda \\
 &= \frac{1}{2} \left\{ \partial_\alpha \partial_\beta g_{\gamma\nu} + \partial_\alpha \partial_\nu g_{\gamma\beta} - \partial_\alpha \partial_\beta g_{\nu\beta} \right. \\
 &\quad \left. - \partial_\beta \partial_\alpha g_{\gamma\nu} - \partial_\beta \partial_\nu g_{\gamma\alpha} + \partial_\nu \partial_\gamma g_{\alpha\beta} \right\} - \Gamma_{\lambda\gamma\alpha} \Gamma_{\nu\beta}^\lambda + \Gamma_{\lambda\gamma\beta} \Gamma_{\nu\alpha}^\lambda \\
 &= \frac{1}{2} \left\{ \partial_\alpha \partial_\nu g_{\gamma\beta} - \partial_\alpha \partial_\beta g_{\gamma\nu} - \partial_\beta \partial_\nu g_{\gamma\alpha} + \partial_\nu \partial_\alpha g_{\beta\alpha} \right\} - g_{\lambda\kappa} \Gamma_{\gamma\alpha}^\kappa \Gamma_{\nu\beta}^\lambda \\
 &\quad + g_{\lambda\kappa} \Gamma_{\gamma\beta}^\kappa \Gamma_{\nu\alpha}^\lambda
 \end{aligned}$$

(17) 12 $\gamma \leftrightarrow \nu$ の λ の換元は 反対称

(17)

$$\Rightarrow R_{\gamma\nu\alpha\beta} = -R_{\nu\gamma\alpha\beta} \quad (18)$$

また、
 $\begin{cases} \gamma \leftrightarrow \alpha \\ \nu \leftrightarrow \beta \end{cases}$ と 対称。

$$R_{\gamma\nu\alpha\beta} = R_{\alpha\beta\gamma\nu} \quad (19)$$

⚠ (12), (13), (18), (19) は $R_{\alpha\beta\gamma\delta}$ の 独立な成分は 20 個

(12), (18) は $(\alpha\beta), (\gamma\delta)$ の 組合せは 4通り ある 6通り ($4^2 = 256$ のうち)

よし (19) は $\frac{6C_2}{2} + 6^{\text{回数}} = 21$ 通り

(13) は $(\alpha, \beta, \gamma, \delta)$ が $(0, 1, 2, 3)$ の 選択条件が 1つ

$21 - 1 = 20$ 個 の 独立

5-4 Bianchi の恒等式 (Bianchi's identity)

- また、1形式 $[\nabla_\alpha, \nabla_\beta]$ と作用させるとどうなるか？

$$[\nabla_\alpha, \nabla_\beta] \omega_\gamma = -R^\lambda_{\gamma\alpha\beta} \omega_\lambda \quad (20)$$

$\left(\begin{array}{l} \\ \\ \end{array} \right)$

$$\begin{aligned} & [\nabla_\alpha, \nabla_\beta] (\omega_\gamma V^\gamma) = 0 \quad (\omega_\gamma V^\gamma \text{ はスカラ}) \\ \Rightarrow & V^\gamma [\nabla_\alpha, \nabla_\beta] \omega_\gamma + \omega_\gamma [\nabla_\alpha, \nabla_\beta] V^\gamma = 0 \quad (\text{Leibnitz}) \\ \Rightarrow & V^\gamma [\nabla_\alpha, \nabla_\beta] \omega_\gamma + \omega_\gamma R^\gamma_{\mu\alpha\beta} V^\mu = 0 \\ V^\gamma \text{ は任意} \Rightarrow & [\nabla_\alpha, \nabla_\beta] \omega_\gamma + R^\lambda_{\gamma\alpha\beta} \omega_\lambda = 0 \end{aligned}$$

- すると、 (1) テンソルであるテンソル積 $U^\nu \omega_\nu$ が出来た。

$$[\nabla_\alpha, \nabla_\beta] (U^\nu \omega_\nu) = R^\mu_{\lambda\alpha\beta} U^\lambda \omega_\mu = R^\lambda_{\nu\alpha\beta} U^\nu \omega_\lambda \quad (21)$$

- (1) $\nabla_\nu V^\mu$ と $[\nabla_\alpha, \nabla_\beta]$ と作用

$$[\nabla_\alpha, \nabla_\beta] \nabla_\nu V^\mu = R^\mu_{\lambda\alpha\beta} \nabla_\nu V^\lambda - R^\lambda_{\nu\alpha\beta} \nabla_\lambda V^\mu \quad (22)$$

- (2) $[\nabla_\alpha, \nabla_\beta] V^\mu$ は ∇_ν と作用

$$\nabla_\nu ([\nabla_\alpha, \nabla_\beta] V^\mu) = \nabla_\nu (R^\mu_{\lambda\alpha\beta} V^\lambda) = \nabla_\nu R^\mu_{\lambda\alpha\beta} \cdot V^\lambda + R^\mu_{\lambda\alpha\beta} \cdot \nabla_\nu V^\lambda \quad (23)$$

- (3) $(23) - (22)$ より

$$[\nabla_\nu, [\nabla_\alpha, \nabla_\beta]] V^\mu = \nabla_\nu R^\mu_{\lambda\alpha\beta} \cdot V^\lambda + R^\lambda_{\nu\alpha\beta} \cdot \nabla_\lambda V^\mu \quad (24)$$

- (4) ④ ν, α, β をサインリックに変えて並べ

$$([\nabla_\nu, [\nabla_\alpha, \nabla_\beta]] + [\nabla_\alpha, [\nabla_\beta, \nabla_\nu]] + [\nabla_\beta, [\nabla_\nu, \nabla_\alpha]]) V^\mu$$

$$= V^\mu (\nabla_\nu R^\mu_{\lambda\alpha\beta} + \nabla_\alpha R^\mu_{\lambda\beta\nu} + \nabla_\beta R^\mu_{\lambda\nu\alpha})$$

$$+ \nabla_\lambda V^\mu (R^\lambda_{\nu\alpha\beta} + R^\lambda_{\alpha\beta\nu} + R^\lambda_{\beta\nu\alpha}) \quad (25)$$

(5) 一般に、演算子 X, Y, Z が交換子

$$[X, Y] \equiv XY - YX = -[Y, X]$$

が定義されるとき。

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad (26)$$

Jacobi の恒等式

(cf. 電子力学の角運動量
Lie 代数)

(6) ⑤ 左辺 = 0 が恒等的成立

右辺第2項は ⑬ か's 0

$$\nabla_\nu R^\mu_{\lambda\alpha\beta} + \nabla_\alpha R^\mu_{\lambda\beta\nu} + \nabla_\beta R^\mu_{\lambda\nu\alpha} = 0 \quad (27)$$

Bianchi の恒等式

5-5 Ricci テンソル, スカラーカー曲率, Einstein テンソル

[Ricci テンソル]

$$R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu} \quad (28)$$

[スカラーカー曲率 (Ricci スカラーカー)]

$$R = R^\mu_r = g_{\mu\nu} R^{\mu\nu} \quad (29)$$

[Einstein テンソル]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (30)$$

△ Ricci テンソル, Einstein テンソルは対称テンソル

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\lambda\alpha} \Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\mu\alpha}$$

上式の 第二項以外は用ひる $\mu \leftrightarrow \nu$ の入れ換えてついで対称,

$$\begin{aligned} \text{第二項: } -\partial_\nu \Gamma^\alpha_{\mu\alpha} &= -\partial_\nu \left[\frac{1}{2} g^{\alpha\kappa} (\partial_\mu g_{\kappa\alpha} + \partial_\alpha g_{\kappa\mu} - \partial_\kappa g_{\mu\alpha}) \right] \\ &\quad \underbrace{\phantom{g^{\alpha\kappa}}}_{\text{打ち消す}} \underbrace{\phantom{g^{\alpha\kappa}}}_{\text{}} \\ &= -\frac{1}{2} \partial_\nu [g^{\alpha\kappa} \partial_\mu g_{\kappa\alpha}] \\ &= -\partial_\nu \partial_\mu \ln \sqrt{-g} \quad (\text{Christoffel 記号の微分の公式より}) \\ &\text{これは } \mu \leftrightarrow \nu \text{ について対称.} \end{aligned}$$

∴ $R_{\mu\nu}$ は対称テンソル

5-6 縮約された Bianchi の恒等式

(27) より

$$\begin{aligned} \nabla_\nu R^\beta_{\lambda\alpha\beta} + \nabla_\alpha R^\beta_{\lambda\beta\nu} + \nabla_\beta R^\beta_{\lambda\nu\alpha} &= 0 \\ -\nabla_\nu R_{\lambda\alpha} + \nabla_\alpha R_{\lambda\nu} + \nabla_\beta R^\beta_{\lambda\nu\alpha} &= 0 \\ \times g^{\lambda\nu} \quad \downarrow & \\ -\nabla_\lambda R^\lambda_\alpha + \nabla_\alpha R + \nabla_\beta R^\beta_{\nu\alpha} &= 0 \end{aligned}$$

$$-\nabla_\lambda R^\lambda_\alpha + \nabla_\alpha R - \nabla_\beta R^\beta_{\nu\alpha} = 0 \quad \left(\nabla_\lambda g_{\mu\nu} = 0 \text{ と } \right)$$

$$-\nabla_\lambda R^\lambda_\alpha + \nabla_\alpha R - \nabla^\beta R^\nu_{\beta\nu\alpha} = 0 \quad \left(\nabla_\lambda g_{\mu\nu} = 0 \text{ と } \right)$$

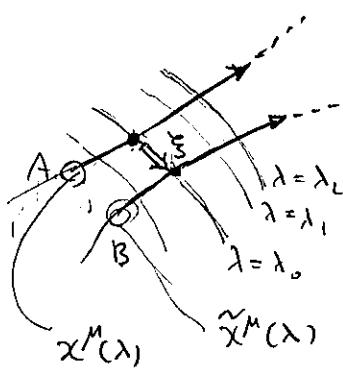
$$-\nabla_\lambda R^\lambda_\alpha + \nabla_\alpha R - \nabla^\beta R^\nu_{\beta\nu\alpha} = 0$$

$$-2\nabla_\lambda R^\lambda_\alpha + \nabla_\alpha R = 0 \Rightarrow \nabla_\lambda (R^\lambda_\alpha - \frac{1}{2} \delta^\lambda_\alpha R) = 0 \quad (31)$$

$$\therefore \nabla_\lambda G^\lambda_\alpha = 0 \quad (32) \quad (G^\mu_\alpha = g^{\mu\lambda} G_{\lambda\alpha} = R^\lambda_\alpha - \frac{1}{2} \delta^\lambda_\alpha R)$$

5-6 測地線偏差の方程式

近接する測地線を延長すると、どう振舞うか調べよう



同じアフィンパラメータ値で共有する 2 つの測地線

$$A: \quad x^{\mu} = x^{\mu}(\tau)$$

$$B: \quad x^{\mu} = \tilde{x}^{\mu}(\tau) = x^{\mu}(\tau) + \xi^{\mu}(\tau)$$

ξ^{μ} は微小量とする。($|\xi| \ll |x|$)

A, B はそれぞれ測地線なので

$$A: \quad \frac{d^2 x^{\mu}}{d\tau^2} + \Gamma_{\nu\lambda}^{\mu}[x^{\alpha}(\tau)] \cdot \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = 0 \quad (33)$$

$$B: \quad \frac{d^2 \tilde{x}^{\mu}}{d\tau^2} + \Gamma_{\nu\lambda}^{\mu}[\tilde{x}^{\alpha}(\tau)] \frac{d\tilde{x}^{\nu}}{d\tau} \frac{d\tilde{x}^{\lambda}}{d\tau} = 0 \quad (34)$$

$$B \Rightarrow \frac{d^2 \tilde{x}^{\mu}}{d\tau^2} + \frac{d^2 \xi^{\mu}}{d\tau^2} + \Gamma_{\nu\lambda}^{\mu}[x^{\alpha} + \xi^{\alpha}] \left(\frac{dx^{\nu}}{d\tau} + \frac{d\xi^{\nu}}{d\tau} \right) \left(\frac{dx^{\lambda}}{d\tau} + \frac{d\xi^{\lambda}}{d\tau} \right) = 0 \quad (35)$$

$$\therefore B - A: \quad \frac{d^2 \xi^{\mu}}{d\tau^2} + \left(\Gamma_{\nu\lambda}^{\mu}[x^{\alpha}] + \partial_{\alpha} \Gamma_{\nu\lambda}^{\mu} \xi^{\alpha} + O(|\xi|^2) \right)$$

$$\times \left(\frac{dx^{\nu}}{d\tau} + \frac{d\xi^{\nu}}{d\tau} \right) \times \left(\frac{dx^{\lambda}}{d\tau} + \frac{d\xi^{\lambda}}{d\tau} \right)$$

$$- \Gamma_{\nu\lambda}^{\mu}[x^{\alpha}] \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = 0 \quad (36)$$

$$\frac{d^2 \xi^{\mu}}{d\tau^2} + \partial_{\alpha} \Gamma_{\nu\lambda}^{\mu} \xi^{\alpha} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} + \underbrace{\Gamma_{\nu\lambda}^{\mu} \frac{dx^{\nu}}{d\tau} \frac{d\xi^{\lambda}}{d\tau}}_{= 0} + \underbrace{\Gamma_{\nu\lambda}^{\mu} \frac{d\xi^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau}}_{= 0} + O(|\xi|^2) = 0$$

$$\Rightarrow \frac{d^2 \xi^{\mu}}{d\tau^2} + \partial_{\alpha} \Gamma_{\nu\lambda}^{\mu} \xi^{\alpha} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} + 2 \Gamma_{\nu\lambda}^{\mu} \frac{dx^{\nu}}{d\tau} \frac{d\xi^{\lambda}}{d\tau} = O(|\xi|^2) \quad (37)$$

二等則地殼運動方程式 $\frac{D}{Dt} V^M$ の微分 $\frac{D}{Dt} \xi^M$

(11)

$$\begin{aligned}\frac{D}{Dt} V^M &= \frac{dx^\alpha}{dt} \nabla_\alpha V^M = \frac{dx^\alpha}{dt} (\partial_\alpha V^M + \Gamma_{\nu\alpha}^\mu V^\nu) \\ &= \frac{d}{dt} V^M + \Gamma_{\nu\alpha}^\mu V^\nu \frac{dx^\alpha}{dt} \quad (38)\end{aligned}$$

を用いて

$$\frac{D}{Dt} \left(\frac{D}{Dt} \xi^M \right) = \frac{D}{Dt} \left(\frac{d}{dt} \xi^M + \Gamma_{\nu\alpha}^\mu V^\nu \frac{dx^\alpha}{dt} \right)$$

$$= \frac{d}{dt} \left(\frac{d}{dt} \xi^M + \Gamma_{\nu\alpha}^\mu \xi^\nu \frac{dx^\alpha}{dt} \right)$$

$$+ \Gamma_{\beta\mu}^\mu \left(\frac{d}{dt} \xi^\beta + \Gamma_{\nu\beta}^\mu \xi^\nu \frac{dx^\alpha}{dt} \right) \cdot \frac{d x^\mu}{dt}$$

$$= \frac{d^2}{dt^2} \xi^M + \frac{d}{dt} \left(\Gamma_{\nu\alpha}^\mu \xi^\nu \frac{dx^\alpha}{dt} \right) + \Gamma_{\rho\mu}^\mu \frac{d \xi^\rho}{dt} \frac{dx^\mu}{dt} + \Gamma_{\rho\mu}^\mu \Gamma_{\nu\alpha}^\rho \xi^\nu \frac{dx^\alpha}{dt} \frac{dx^\mu}{dt}$$

$$= \frac{d^2}{dt^2} \xi^M + \left(\frac{d}{dt} \Gamma_{\nu\alpha}^\mu \right) \xi^\nu \frac{dx^\alpha}{dt} + \Gamma_{\nu\alpha}^\mu \frac{d \xi^\nu}{dt} \frac{dx^\alpha}{dt} + \Gamma_{\nu\alpha}^\mu \xi^\nu \frac{d^2 x^\alpha}{dt^2}$$

$$(37) = \boxed{\frac{d^2}{dt^2} \xi^M + \partial_\alpha \Gamma_{\nu\lambda}^\mu \frac{dx^\lambda}{dt} \xi^\nu \frac{dx^\alpha}{dt} + \boxed{2 \Gamma_{\nu\lambda}^\mu \xi^\nu \frac{dx^\lambda}{dt} \frac{dx^\mu}{dt}}}$$

$$- \Gamma_{\nu\alpha}^\mu \xi^\nu \Gamma_{\mu\lambda}^\lambda \frac{dx^\alpha}{dt} \frac{dx^\lambda}{dt} + \Gamma_{\rho\mu}^\mu \Gamma_{\nu\alpha}^\rho \xi^\nu \frac{dx^\alpha}{dt} \frac{dx^\mu}{dt}$$

$$= \boxed{\underbrace{\partial_\alpha \Gamma_{\nu\lambda}^\mu - \partial_\nu \Gamma_{\alpha\lambda}^\mu + \Gamma_{\rho\mu}^\mu \Gamma_{\lambda\nu}^\rho - \Gamma_{\nu\beta}^\mu \Gamma_{\lambda\alpha}^\beta}_{R^\mu_{\lambda\nu\alpha}} \frac{dx^\alpha}{dt} \xi^\nu \frac{dx^\lambda}{dt}}$$

$$= \boxed{- R^\mu_{\lambda\nu\alpha} \frac{dx^\alpha}{dt} \xi^\nu \frac{dx^\lambda}{dt}} \quad (39)$$

$$\therefore \frac{D^2}{Dt^2} \xi^M + R^\mu_{\lambda\nu\alpha} \frac{dx^\alpha}{dt} \xi^\nu \frac{dx^\lambda}{dt} = 0. \quad (40)$$

運動方程式