Why is the radius of a neutron star $\mathcal{O}[10^{-3}]$ times smaller than that of a white dwarf?

Shin Yoshida*

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It is known that a typical radius of a neutron star with a mass of $1M_\odot$[1] is 10^6cm, while a white dwarf with the same mass is typically in a size of the Earth ($\sim 10^9$cm). Below we derive a simple scaling relation between the relative mass of degenerate Fermion to the nucleon mass μ, which dominates the pressure of stellar matter, and the radius of the star R,

$$R \propto \mu^{-1}. \tag{1}$$

Since $\mu \sim 0.5 \times 10^{-3}$ for electron, we see a neutron star is $\mathcal{O}[10^{-3}]$ smaller than a white dwarf with the same mass.

The hydrostatic equation of a star is written as

$$\frac{1}{\rho}\frac{dP}{dr} + \frac{d\Phi}{dr} = 0, \tag{2}$$

where ρ, P, Φ are mass density, pressure and gravitational potential. Approximating the derivative in evaluating the equation at the stellar surface as,

$$\frac{1}{\rho}\frac{dP}{dr} \sim \frac{1}{\bar{\rho}} \cdot \frac{0 - P}{R}, \tag{3}$$

and

$$\frac{d\Phi}{dr} = \frac{GM}{R^2}. \tag{4}$$

Here $\bar{\rho}$ is the average density of the star. Thus,

$$P = \bar{\rho}\frac{GM}{R}. \tag{5}$$

The definition of average density is

$$\bar{\rho} \cdot \frac{4\pi}{3} R^3 = M. \tag{6}$$

*Department of Earth Science and Astronomy, Graduate School of Arts and Sciences, The University of Tokyo

[1] $1M_\odot = 1.989 \times 10^{33}$g.

We consider a perfectly degenerate gas of Femion with mass m dominates the gas pressure. Number density n of the Fermion is related to its Fermi momentum $p_{_F}$ as,

$$n = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2 dp = \frac{8\pi}{3h^3}p_{_F}^3, \tag{7}$$

where h is the Planck constant.

Mass density of the gas is defined as,

$$\bar{\rho} = m_{_N} n, \tag{8}$$

where $m_{_N}$ is the mass of a nucleon. In the case of electron degeneracy, n is that of electron and Eq.(8) is strictly valid only when the gas is composed of pure hydrogen.

Gas pressure P is dominated by the Fermion and is written as,

$$\begin{aligned} P &= \frac{1}{3}\frac{2}{h^3}\int_0^{p_F} 4\pi p^2 dp\, pv = \frac{2}{3h^3}\int_0^{p_F} 4\pi p^2 dp \frac{p^2}{m} \\ &= \frac{8\pi}{15h^3}\frac{p_{_F}^5}{m}, \end{aligned} \tag{9}$$

where v is the velocity of the Fermion which we assume to be non-relativistic.

From Eq.(5) and (6) we have,

$$P = \frac{3G}{4\pi}\frac{M^2}{R^4}, \tag{10}$$

and

$$\bar{\rho} = \frac{3}{4\pi}\frac{M}{R^3}. \tag{11}$$

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From Eq.(7), (8) and (9), we have

$$\left(\frac{15h^3}{8\pi} mP\right)^3 = \left(\frac{3h^3}{8\pi}\frac{\bar{\rho}}{m_{_N}}\right)^5. \tag{12}$$

Using Eq.(10) and (11), the equation above is cast into

$$\left(\frac{3G}{4\pi}\frac{mM^2}{R^4}\right)^3 = \frac{1}{125}\left(\frac{3h^3}{8\pi}\right)^2\left(\frac{3M}{4\pi}\frac{1}{R^3 m_{_N}}\right)^5. \tag{13}$$

Introducing $\mu \equiv m/m_{_N}$, The equation is written as,

$$\left(\frac{3G}{4\pi}\right)^3 m_{_N}^8 M(\mu R)^3 = \frac{1}{125}\left(\frac{3h^3}{8\pi}\right)^2, \tag{14}$$

thus

$$\mu R \propto M^{-\frac{1}{3}}. \tag{15}$$

For constant M, we see the radius R is inversely proportional to μ, the ratio of the Fermion mass to the nucleon mass. For electron, we have $\mu = 5.45 \times 10^{-4}$. Therefore we have the radius of a white dwarf radius is $\mathcal{O}[10^3]$ larger than that of a neutron star (for which $\mu = 1$) with a same mass. [2]

[2] It should be noted that nuclear force (mostly repulsive) is not negligible in a neutron star, which may largely modify the gas pressure of degenerate neutrons.