Dummy's note: (1) 'Bernoulli'?

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In some literature on numerical construction of equilibrium gaseous masses, Bernoulli equation is mistakenly refered to. This is intended to clarify this issue. We work here in a framework of Newtonian fluid dynamics.

1 Bernoulli equation

Equation of motion of fluid is written as,

$$(\partial_t + \pounds_{\bar{v}})\tilde{v} + \frac{1}{\rho}\tilde{d}p + \tilde{d}\left(\Phi - \frac{1}{2}v^2\right) = 0, \tag{1}$$

where $\bar{v}$ and $\tilde{v}$ mean velocity vector and its dual 1-form. Φ is gravitational poteintial. Note that v^2 is abbreviation of scalar $<\bar{v}, \tilde{v}>= \bar{v}^i\tilde{v}_i$. For simplicity, we assume barotropicity of fluid, $p = p(\rho)$, then,

$$(\partial_t + \pounds_{\bar{v}})\tilde{v} + \tilde{d}\left(h + \Phi - \frac{1}{2}v^2\right) = 0, \tag{2}$$

where $h = \int dp/\rho$ is enthalpy.

Assuming steady flow $\partial_t\bar{v} = 0$, we multiply this equation with $\bar{v}$ to get,

$$<\bar{v}, \pounds_{\bar{v}}\tilde{v}> + <\bar{v}, \tilde{d}\left(h + \Phi - \frac{1}{2}v^2\right)>= 0. \tag{3}$$

Taking into account the identity $\pounds_{\bar{v}}v^2 = \pounds_{\bar{v}} <\bar{v}, \tilde{v}>=<\bar{v}, \pounds_{\bar{v}}\tilde{v}> + <\pounds_{\bar{v}}\bar{v}, \tilde{v}>$ and $\pounds_{\bar{v}}\bar{v} = 0$, we get,

$$\bar{v}\left[h + \Phi + \frac{1}{2}v^2\right] = 0. \tag{4}$$

This says that the quantity in the bracket is constant along flow lines (lines tangent to flow vector).

1.1 A first integral of Euler equation – not Bernoulli

We again assume steady flow $\partial_t\bar{v} = 0$. Further we assume a flow is circular, or more generally Lie-dragged by itself: $\pounds_{\bar{v}}\tilde{v} = 0$. In that case Eq.(2) leads to,

$$\tilde{d}\left(h + \Phi - \frac{1}{2}v^2\right) = 0, \quad \text{thus,} \quad h + \Phi - \frac{1}{2}v^2 = C \text{ (const.)}. \tag{5}$$

The constancy here applies throughout the entire fluid.