Dummy's note (4): Mechanical toy models of quasi-normal mode of radiating system

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Abstract

Here we gather 4 toy models that mimick quasi-normal mode of relativistic system.

1 Mimicking black hole QNM (Fig.1)

We here introduce a simplest toy model of black hole quasinormal mode (QNM). I'm not quite sure, but this might have been first solved in [1]. The model has an infinitly stretched string with constant

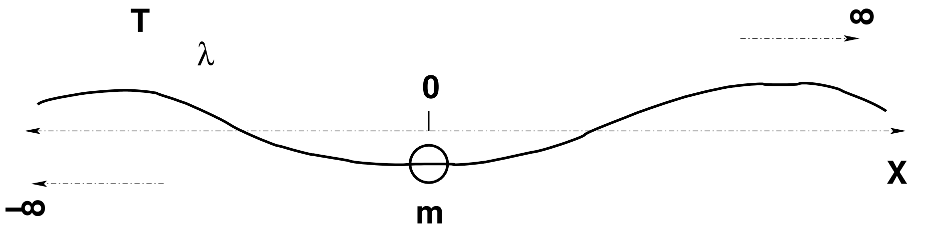

Figure 1: Schematic figure of toy model for quasi-normal mode of a black hole.

tension T and constant linear mass density λ_0. We attach a point mass at $x = 0$. Then linear mass density of string is

$$\lambda = \lambda_0 + m\delta(x). \tag{1}$$

The sound velocity for $x \neq 0$ is $c^2 = T/\lambda_0$

We assume harmonic dependence of displacement $\eta(x,t) = y(x)e^{-i\omega t}$.

The boundary condition of free oscillation is 1) no incoming wave from $x \to \infty$ and 2) no incoming wave from $x \to -\infty$. The first condition gives,

$$\frac{\partial \eta}{\partial x} - i\frac{\omega}{c}\eta\Big|_{x\to\infty} = 0, \tag{2}$$

and the second one,

$$\frac{\partial \eta}{\partial x} + i\frac{\omega}{c}\eta\Big|_{x\to-\infty} = 0. \tag{3}$$

From the harmonic decomposition ansatz, we have

$$\frac{d^2 y}{dx^2} + \omega^2\frac{\lambda}{T}y = 0. \tag{4}$$

By using (2) and (3) we have

$$y_L = ae^{-i\omega x}, \quad (x < 0) \tag{5}$$
$$y_R = be^{i\omega x}, \quad (x > 0). \tag{6}$$

Around $x = 0$ we integrate (4) to obtain one of 2 junction conditions at x=0,

$$\int_{-\epsilon}^{epsilon} \left(\frac{d^2y}{dx^2} + \omega^2 \frac{\lambda}{T} y \right) dx = 0 \tag{7}$$

thus

$$\left[\frac{dy}{dx} \right]_{-0}^{+0} + \omega^2 \frac{m}{T} y(0) = 0. \tag{8}$$

Another condition is the continuity of string

$$y_L(-0) = y_R(+0). \tag{9}$$

From two conditions above we have a dispersion relation. From (9), we have

$$b = a, \tag{10}$$

while from (8), we have

$$i\frac{\omega}{c} b - \left(-i\frac{\omega}{c} a \right) + \omega^2 \frac{m}{T} a = 0. \tag{11}$$

These gives us a simple dispersion relation

$$\omega^2 \frac{m}{T} + 2i\frac{\omega}{c} = 0. \tag{12}$$

This equation gives solution $\omega = 0, -i\frac{2T}{mc}$. The first one corresponds to a trivial parallel shift of string. The other corresponds to a fast damping quasi-normal mode.

2 Mimicking spacetime modes - A (Fig.2)

Here we introduce a toy model of spacetime mode in relativistic stars. Prepare a half-infinite string with constant tension T and linear mass density λ_0. Attach a point mass m at $x = L$. The system is different from that in the former section in that the string is fixed at $x = 0$. Then the basic equation

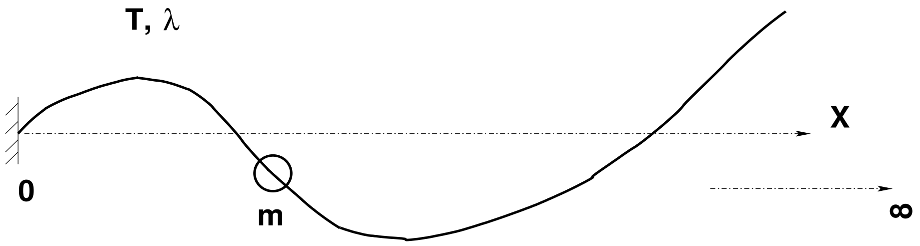

Figure 2: Schematic figure of toy model for spacetime mode of a star. Case (A).

is almost equilvalent to those in the former section except the boundary condition at $x = 0$

$$\eta(x = 0; t) = 0. \tag{13}$$

We can follow the same procedure as before and obtain a dispersion relation

$$\frac{e^\theta + e^{-\theta}}{e^\theta + e^{-\theta}} = 1 - \left(\frac{mc^2}{TL} \right) \theta, \tag{14}$$

where $\theta := \frac{i\omega L}{c}$ and $c^2 := T/\lambda_0$

3 Mimicking spacetime modes - B (Fig.3)

Here we consider another toy model for spacetime mode of relativistic star. It consists of half-infinite string fixed at $x = 0$ with constant tension T. Linear mass density is a function of position. For simplicity we take

$$\begin{aligned} \lambda &= \lambda_1 \quad (0 \le x \le L) \\ &= \lambda_2 \quad (L \le x). \end{aligned} \tag{15}$$

We define sound velocity in the each portion of string as

$$c_1^2 = \frac{T}{\lambda_1}, \quad c_2^2 = \frac{T}{\lambda_2} \tag{16}$$

Boundary conditions are 1) displacement η is zero at $x = 0$ and 2) wave is outgoing at $x \to$

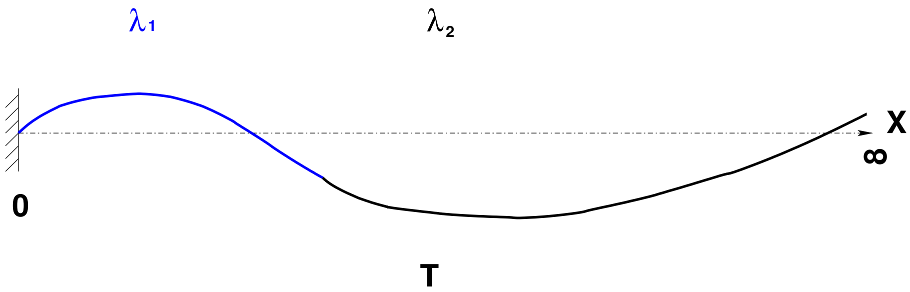

Figure 3: Schematic figure of toy model for quasi-normal mode of a star. Case (B)

∞. Considering these condition and take into account junction conditions at $x = L$, which are 3) displacement is continuous ($\eta(L-0) = \eta(L+0)$) and 4) spatial derivatite of η satisfies jump condition obtained by integrating wave equation in $x : [L-0, L+0]$, we obtain dispersion relation of this model

$$\frac{e^\zeta + 1}{e^\zeta - 1} = \beta \tag{17}$$

where $\beta = c_1/c_2$ and $\zeta = 2i\omega L/c_1$.

For an example we compute a quasi-normal mode for $\beta = 1/2$ case. In this case we have $\zeta = 1.09861 \pm i\pi$. Then

$$\omega = \frac{c_1}{2L}(\pm\pi - 1.09861i). \tag{18}$$

and the mode damps in the same timescale as the oscillation.

4 Mimicking coupled modes of a star and spacetime (see Fig.4)

In this section we treat a toy model of star coupled with gravitational field. We modify the model introduced by [2], in which both stars and spacetime are mimicked by string with a constant line mass density. Here we mimick the star with a single spring (thus our 'star' has only one natural frequency), while the string has mass density as a function of position, so that we also have analogue of trapped spacetime mode (w-mode).

4.1 Master equation

In this case fluid motion of star is modeled by a single spring with spring constant k to which a mass m is attached. Spacetime around the star is modeled by a half-infinite string with line mass density $\lambda(x)$ (coordinate x is set along equilirbrium string, whose origin is the left end of the string.). The tension in the string is constant T. Two of the oscillating system above is coupled by a weak

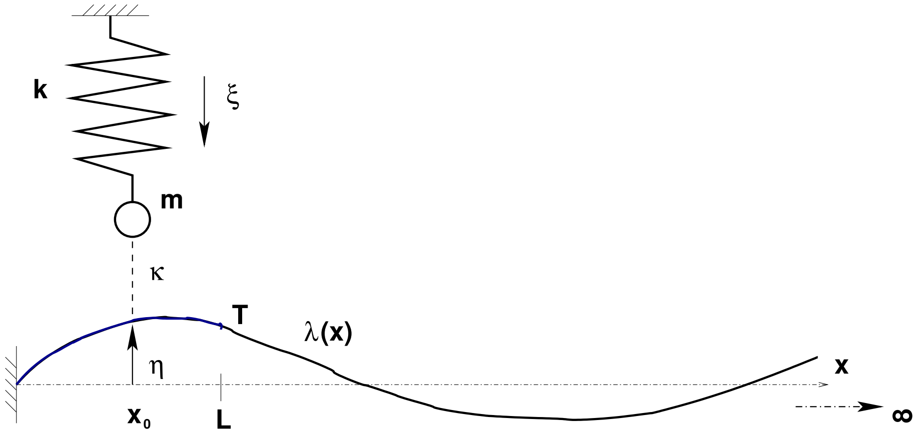

Figure 4: Schematic figure of toy model for quasi-normal mode of a stellar oscillation coupled to spacetime one.

coupler κ which enables transmission of energy between the string and spring. We define downward displacement of spring as $\xi(t)$ and upward displacement of string as $\eta(x,t)$. By introducing natural frequency of spring ω_0, we can describe the coupled system by

$$\frac{d^2}{dt^2}\xi + \omega_0^2 = -\frac{\kappa}{m}\left(\xi + \eta(x_0,t)\right) \tag{19}$$

$$\lambda\frac{\partial^2}{\partial t^2}\eta = T\frac{\partial^2}{\partial x^2}\eta - \kappa\left(\eta(x,t) + \xi\right)\delta(x - x_0). \tag{20}$$

In the discussion below we simplify the line mass density of string as

$$\lambda(x) = \begin{array}{ll} \lambda_1 & (0 \le x \le L) \\ \lambda_2 & (L \le x), \end{array} \tag{21}$$

where λ_1 and λ_2 are constant with $\lambda_1 < \lambda_2$. Corresponding sound speeds are defined as $c_i^2 := T/\lambda_i$ $(i = 1, 2)$. We consider the case in which the coupler is attached to string at $x = x_0$ that satisfies $x_0 < L$.

For both ξ and η, we assume harmonic dependence and write them as $(\xi, \eta) = (\xi_\omega, \eta_\omega(x))e^{-i\omega t}$ (ξ_ω is a constant). Then equations above reduce to

$$(-\omega^2 + \omega_0^2)\xi_\omega = -\kappa_A(\xi_\omega + \eta_\omega(x)) \tag{22}$$

$$(\omega^2 + c_i^2)\frac{d^2}{dx^2}\eta_\omega = \kappa_B(\eta_\omega(x) + \xi_\omega)\delta(x - x_0) \tag{23}$$

where $\kappa_A := \kappa/m$ and $\kappa_B := \kappa/\lambda_1$.

Substituting Eq.(22) to Eq.(23) we obtain

$$\left(\omega^2 + c_i^2\frac{d^2}{dx^2}\right)\eta_\omega = \kappa_B\left(\eta_\omega(x) + \frac{\kappa_A}{\omega^2 - \omega_0^2 - \kappa_A}\eta_\omega(x_0)\right)\delta(x - x_0). \tag{24}$$

4.2 Boundary and junction condition

4.2.1 Boundary conditions

Boundary conditions for η_ω are set at $x = 0$ and $x \to \infty$. At $x = 0$ the string is fixed thus

$$\eta_\omega(x = 0, t) = 0. \tag{25}$$

For $x \to \infty$ outgoing boundary condition is imposed

$$\frac{d\eta_\omega}{dx} - i\frac{\omega}{c_2}\eta_\omega \Big|_{x\to\infty} = 0 \tag{26}$$

By imposing Eq.(25), we obtain "left" solution valid in $0 \le x \le x_0$

$$\eta_\omega^\ell = \alpha \sin \frac{\omega}{c_1} x, \tag{27}$$

where α is constant. While Eq.(26) gives

$$\eta_\omega^r = \beta e^{i\frac{\omega}{c_2}x} \tag{28}$$

4.2.2 Junction condition

At $x = x_0$ two junction conditions must be satisfied. One is the continuity of displacement

$$\eta_\omega(x_0 - 0) = \eta_\omega(x_0 + 0). \tag{29}$$

To obtain the condition for first derivative of η_ω, we integrate Eq.(24) in $[x_0 - 0, x_0 + 0)$

$$c_1^2 \left[\frac{d\eta_\omega}{dx}\right]_{x_0-0}^{x_0+0} = \kappa_B \left(\eta_\omega(x_0) + \frac{\kappa_A}{\omega^2 - \omega_0^2 - \kappa_A}\eta_\omega(x_0)\right). \tag{30}$$

To write down this condition we introduce "middle" solution valid in $x_0 \le xleL$

$$\eta_\omega^m = \gamma e^{-i\frac{\omega}{c_1}x} + \psi e^{i\frac{\omega}{c_1}x}, \tag{31}$$

where γ, ψ are constant. Then the junction condition above are reduced to be

$$\alpha \sin \frac{\omega}{c_1} x_0 = \gamma e^{-i\frac{\omega}{c_1}x_0} + \psi e^{i\frac{\omega}{c_1}x_0}, \tag{32}$$

$$\begin{aligned} & c_1^2 \left(-i\left(\frac{\omega}{c_1}\right)\gamma e^{-i\frac{\omega}{c_1}x_0} + \left(i\frac{\omega}{c_1}\right)\psi e^{i\frac{\omega}{c_1}x_0} - \alpha\left(\frac{\omega}{c_1}\right)\cos\frac{\omega}{c_1}x_0\right) \\ = \quad & \frac{\kappa_B(\omega^2 - \omega_0^2)}{\omega^2 - \omega_0^2 - \kappa_A}\alpha \sin\frac{\omega}{c_1}x_0 \end{aligned} \tag{33}$$

4.2.3 Junction at $x = L$

Junction conditions at $x = L$ are smooth connection of η_ω

$$\eta_\omega^m(L - 0) = \eta_\omega^r(L + 0), \quad \frac{d\eta_\omega^m}{dx}(L - 0) = \frac{d\eta_\omega^r}{dx}(L + 0). \tag{34}$$

These reduce to the following equations

$$\gamma e^{-i\frac{\omega}{c_1}L} + \psi e^{i\frac{\omega}{c_1}L} = \beta e^{i\frac{\omega}{c_2}L} \tag{35}$$

$$\left(-i\frac{\omega}{c_1}\right)\gamma e^{-i\frac{\omega}{c_1}L} + \left(i\frac{\omega}{c_1}\right)\psi e^{i\frac{\omega}{c_1}L} = \left(i\frac{\omega}{c_2}\right)\beta e^{i\frac{\omega}{c_2}L} \tag{36}$$

Eqs.(32,33, 35,36) gives dispersion relation.

4.3 Dispersion equation

From Eqs.(32,33) we have

$$\begin{aligned}&c_1^2\left(\left(-i\frac{\omega}{c_1}\right)e^{-i\frac{\omega}{c_1}x_0}\gamma\left(i\frac{\omega}{c_1}\right)e^{i\frac{\omega}{c_1}x_0}\psi-\frac{\omega}{2c_1}(e^{i\frac{\omega}{c_1}x_0}+e^{-i\frac{\omega}{c_1}x_0})\alpha\right)\\&=\frac{\kappa_B(\omega^2-\omega_0^2)}{\omega^2-\omega_0^2-\kappa_A}(e^{-i\frac{\omega}{c_1}x_0}\gamma+e^{-i\frac{\omega}{c_1}x_0}\psi),\end{aligned}\tag{37}$$

while Eq.(32) reduces to

$$\frac{1}{2i}\left(e^{i\frac{\omega}{c_1}x_0}-e^{i\frac{\omega}{c_1}x_0}\right)\alpha=e^{-i\frac{\omega}{c_1}x_0}\gamma+e^{i\frac{\omega}{c_1}x_0}\psi.\tag{38}$$

From Eqs.(35,36)

$$-i\frac{\omega}{c_1}e^{-i\frac{\omega}{c_1}L}\gamma+i\frac{\omega}{c_1}e^{i\frac{\omega}{c_1}L}\psi=i\frac{\omega}{c_2}\left(e^{-i\frac{\omega}{c_1}L}\gamma+e^{i\frac{\omega}{c_1}L}\psi\right)\tag{39}$$

For simplicity of expressions below, we introduce

$$e^{i\frac{\omega}{c_1}x_0}:=A,\quad e^{i\frac{\omega}{c_1}L}:=B.\tag{40}$$

Then Eq.(37) is written as

$$\left(\frac{A^2+1}{2}\right)\alpha+\left(i+\frac{\frac{\kappa_B}{c_1}(\omega^2-\omega_0^2)}{\omega(\omega^2-\omega_0^2-\kappa_A)}\right)\gamma+A^2\left(-i+\frac{\frac{\kappa_B}{c_1}(\omega^2-\omega_0^2)}{\omega(\omega^2-\omega_0^2-\kappa_A)}\right)\psi=0\tag{41}$$

From Eq.(38) we have

$$(A^2-1)\alpha-2i\gamma-2iA^2\psi=0.\tag{42}$$

We can simplify Eq.(39) as

$$\gamma=\frac{c_2-c_1}{c_2+c_1}B^2\psi\tag{43}$$

By using Eq.(43) to Eq.(41) we obtain

$$\begin{aligned}0\;=\;&\left(\frac{A^2+1}{2}\right)\alpha\\&+\left(\left(i+\frac{\frac{\kappa_B}{c_1}(\omega^2-\omega_0^2)}{\omega(\omega^2-\omega_0^2-\kappa_A)}\right)\left(\frac{c_2-c_1}{c_2+c_1}\right)B^2+A^2\left(-i+\frac{\frac{\kappa_B}{c_1}(\omega^2-\omega_0^2)}{\omega(\omega^2-\omega_0^2-\kappa_A)}\right)\right)\psi.\end{aligned}\tag{44}$$

From Eq.(43) and Eq.(42) we have

$$(A^2-1)\alpha-2i\left(\left(\frac{c_2-c_1}{c_2+c_1}\right)B^2+A^2\right)\psi=0\tag{45}$$

Finally dispersion equation is obtained by setting determinant of coefficient matrix of Eqs.(44,45)

$$\begin{aligned}&(A^2-1)\left(\tilde{\omega}(\tilde{\omega}^2-\tilde{\omega_0}^2-\tilde{\kappa}_A)-i\tilde{\kappa}_B(\tilde{\omega}^2-1)\left(\frac{1-\beta}{1+\beta}\right)B^2-\left(\tilde{\omega}(\tilde{\omega}^2-\tilde{\omega_0}^2-\tilde{\kappa}_A)+i\tilde{\kappa}_B(\tilde{\omega}^2-1)\right)A^2\right)\\&+\tilde{\omega}(\tilde{\omega}^2-\tilde{\omega_0}^2-\tilde{\kappa}_A)(A^2+1)\left(\left(\frac{1-\beta}{1+\beta}\right)B^2+A^2\right)=0.\end{aligned}\tag{46}$$

Here we introduced dimensionless parameters as

$$\beta:=\frac{c_1}{c_2},\ \tilde{\omega}:=\frac{\omega}{\omega_0},\ \tilde{\kappa}_A:=\frac{\kappa_A}{\omega_0^2},\ \tilde{\kappa}_B:=\frac{\kappa_B}{c_1\omega_0}\tag{47}$$

4.4 Some numerical results

As we expect, we see two kinds of distinct modes in the system. One is the mode which corresponds to spring oscillation (whose natural frequency is normalized to be unity) damped by the coupling to the string. This is an analogue of stellar fluid mode coupled to gravitational radiation. The other is the fast damping mode of string that gives analogue of spacetime mode that is weakly trapped in the curvature of background spacetime.

For parameter set

$$\beta = 0.2,\ \tilde{\kappa}_A = 10^{-3} = \tilde{\kappa}_B,\ \frac{\omega_0 x_0}{\beta c_2} := \tilde{\tau} = 0.4,\ \frac{\omega_0 L}{\beta c_2} := \tilde{T} = 0.8, \tag{48}$$

we obtain for the former class frequency,

$$\tilde{\omega} = 1.0005 - i \cdot 2.99616 \times 10^{-8} \tag{49}$$

while the latter

$$\tilde{\omega} = 1.96382 - i \cdot 0.253439. \tag{50}$$

Note that in non-coupling limit the former frequency becomes $\tilde{\omega} = 1$ and the latter $\tilde{\omega} = 1.9635 - i \cdot 0.253416$.

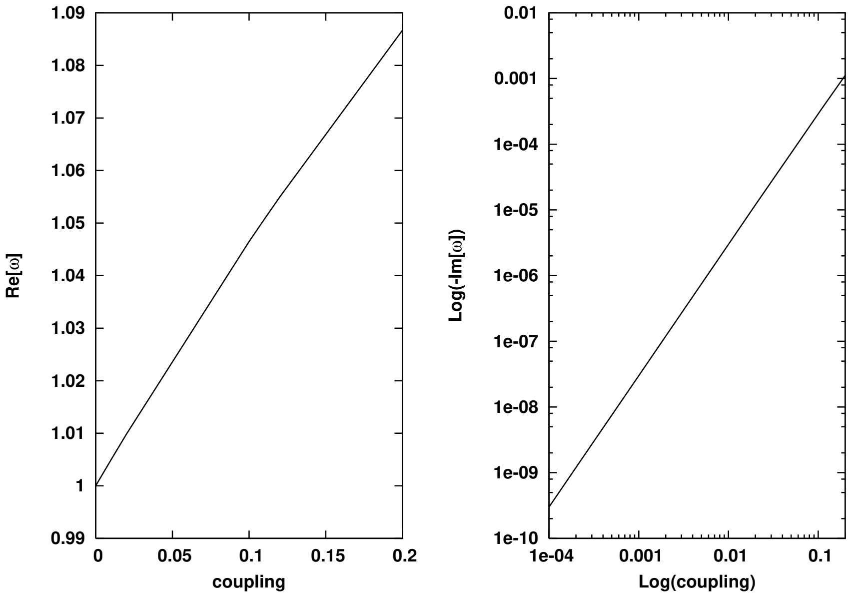

Figure 5: Real and imaginary part of eigenmode that corresponds to the spring oscillation in the limit of coupling goes to zero.

References

[1] Kokkotas, K.D., MSc thesis, University of Wales College of Cardiff, unpublished (1985)

[2] Kokkotas, K.D., Schutz B.F., Gen. Rel. Grav., 18, 913 (1986)

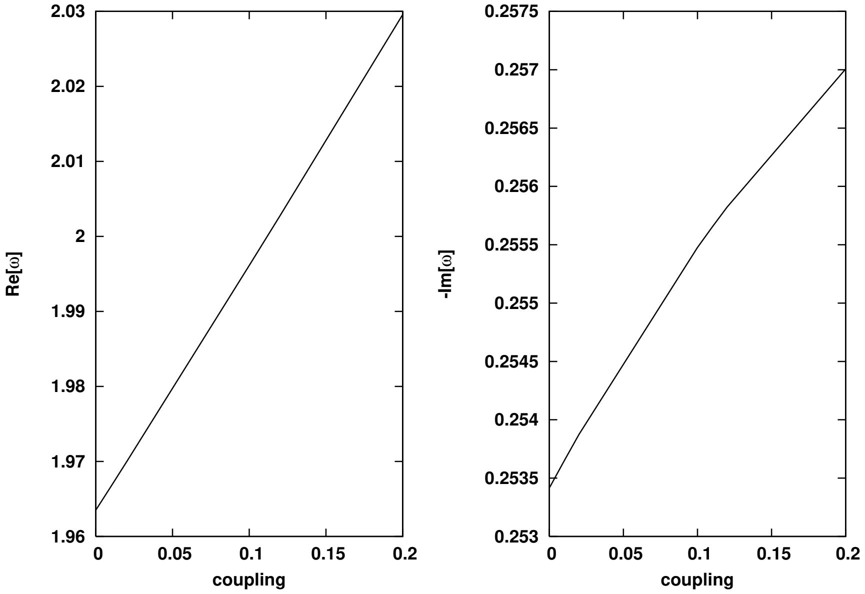

Figure 6: Real and imaginary part of eigenmode that corresponds to the string oscillation in the limit of coupling goes to zero.