Extracting Regge-Wheeler potentials of perturbation to Schwarzschild metric

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October 11, 2006

For our current interest, it is sufficient to consider polar perturbation. Thus we omit axial part of perturbation in the following.

1 Background metric

Assume background metric is written in Schwarzschild coordinate

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega \qquad (1)\\ &= -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega. \qquad (2)\end{aligned}$$

Nonzero connection coefficients are following

$$\begin{aligned} &\Gamma^t_{tr} = \frac{1}{2}\partial_r \ln e^{2\nu},\\ &\Gamma^r_{tt} = \frac{1}{2}e^{-2\lambda}\partial_r e^{2\nu},\ \Gamma^r_{tt} = \frac{1}{2}\partial_r \ln e^{2\lambda},\\ &\Gamma^r_{\theta\theta} = -re^{-2\lambda}, \Gamma^r_{\varphi\varphi} = -e^{-2\lambda} r\sin^2\theta,\\ &\Gamma^\theta_{r\theta} = \frac{1}{r},\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\ \cos\theta,\\ &\Gamma^\varphi_{r\varphi} = \frac{1}{r}, \Gamma^\varphi_{\theta\varphi} = \cot\theta \qquad (3)\end{aligned}$$

2 Tensor spherical harmonics

We follow the definition of tensor spherical harmonics in [1].

2-scalar quantity (e.g. gas pressure, tt, tr and rr components of 4-metric $g_{\mu\nu}$, etc.) is expanded by usual scalar spherical harmonics Y_{lm}. Polar vector (actually 1-form here) is expanded by a basis

$$\tilde{V}_P = D_a Y_{lm}\ \tilde{d}x^a = \partial_\theta Y_{lm}\ \tilde{d}\theta + \partial_\varphi Y_{lm}\ \tilde{d}\varphi. \qquad (4)$$

Here D_a means covariant derivative with respect to 2-metric σ_{ab} on the surface of sphere with unit radius. $\sigma = \mathrm{diag}[1, \sin^2\theta]$. Latin index means $a = \theta, \varphi$. [1]

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[1] Another fundamental quantity on the surface to be used in tensor spherical harmonics is antisymmetric tensor

$$\epsilon_{ab} = \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (5)$$

but we do not need it for the analysis of polar perturbation.

There are two polar tensor basis. They are defined as

$$\overset{1}{T}_{lm} = D_b D_a Y_{lm}\ \tilde{d}x^a \otimes \tilde{d}x^b \tag{6}$$

$$= \partial_\theta^2 Y_{lm}\ \tilde{d}\theta \otimes \tilde{d}\theta + (\partial_\varphi \partial_\theta Y_{lm} - \cot\theta \partial_\varphi Y_{lm}) \cdot (\tilde{d}\theta \otimes \tilde{d}\varphi + \tilde{d}\varphi \otimes \tilde{d}\theta) + (\partial_\varphi^2 Y_{lm} + \sin\theta\cos\theta \partial_\theta Y_{lm}) \tilde{d}\varphi \otimes \tilde{d}\varphi \tag{7}$$

and

$$\overset{2}{T}_{lm} = Y_{lm} \sigma_{ab} \tilde{d}x^a \otimes \tilde{d}x^b \tag{8}$$

$$= Y_{lm} \tilde{d}\theta \otimes \tilde{d}\theta + Y_{lm} \sin^2\theta \tilde{d}\varphi \otimes \tilde{d}\varphi. \tag{9}$$

3 Polar gauge vector and tensor

In general we can write infinitesimal gauge vector η_μ of polar perturbation for order l and degree m as

$$\tilde{\eta} = u_0(t,r) Y_{lm} \tilde{d}t + u_1(t,r) Y_{lm} \tilde{d}r + u_2(t,r) \partial_\theta Y_{lm} \tilde{d}\theta + u_2(t,r) \partial_\varphi Y_{lm} \tilde{d}\varphi. \tag{10}$$

Here 3 functions u_0, u_1 and u_2 are used to fix gauge of perturbation.

Using this gauge vector, an infinitesimal transformation of 4-metric is written as

$$\delta g'_{\mu\nu} = \delta g_{\mu\nu} + 2\nabla_{(\mu} \eta_{\nu)}. \tag{11}$$

Computation of gauge transformation tensor is straightforward. The result is

$$\begin{aligned} \eta_{(\mu;\nu)} \tilde{d}x^\mu \otimes \tilde{d}x^\nu = {} & \frac{1}{2}(\partial_t u_0 - \Gamma^r_{tt} u_1) Y_{lm} \tilde{d}t \otimes \tilde{d}t \\ & + \frac{1}{2}(\partial_r u_0 + \partial_t u_1 - \Gamma^t_{tr} u_0) Y_{lm} (\tilde{d}t \otimes \tilde{d}r + \tilde{d}r \otimes \tilde{d}t) \\ & + (\tilde{w} \otimes \tilde{V}_P + \tilde{V}_P \otimes \tilde{w}) \\ & + u_2 \cdot \overset{1}{T}_{lm} \\ & + r e^{-2\lambda} u_1 \cdot \overset{2}{T}_{lm}, \end{aligned} \tag{12}$$

where we define 1-form $\tilde{w}$ as

$$\tilde{w} = \frac{1}{2}(u_0 + \partial_t u_2) \tilde{d}t + \frac{1}{2}\left(u_1 + \partial_r u_2 - \frac{u_2}{r}\right) \tilde{d}r. \tag{13}$$

4 Transformation to Regge-Wheeler gauge

Suppose we have a general (polar) metric perturbation on Schwarzshild coordinate as $g_{\mu\nu} = \overset{0}{g}_{\mu\nu} + h_{\mu\nu}$. Here $\overset{0}{g}_{\mu\nu}$ is background Schwarzshild metric.

General perturbation $h_{\mu\nu}$ is decomposed by polar tensor harmonics as (for simplicity, we write on a single harmonic (l,m))

$$\begin{aligned} h_{\mu\nu} \tilde{d}x^\mu \otimes \tilde{d}x^\nu = {} & H_0(t,r) Y_{lm} \tilde{d}t \otimes \tilde{d}t \\ & + H_1(t,r) Y_{lm} (\tilde{d}t \otimes \tilde{d}r + \tilde{d}r \otimes \tilde{d}t) \\ & + h_0(t,r)(\tilde{d}t \otimes \tilde{V}_P + \tilde{V}_P \otimes \tilde{d}t) \\ & + h_1(t,r)(\tilde{d}r \otimes \tilde{V}_P + \tilde{V}_P \otimes \tilde{d}r) \\ & + r^2 G(t,r) \cdot \overset{1}{T}_{lm} \\ & + r^2 K(t,r) \cdot \overset{2}{T}_{lm}. \end{aligned} \tag{14}$$

The idea of Regge and Wheeler [2] is to nullify the higher order derivative terms with respect to angular coordinate by using the gauge function u_0, u_1 and u_2. This is done in the following way. The highest derivatives appear in the basis tensor $\overset{1}{T}_{lm}$. Thus we nullify this term (G) by adding $u_2(t,r)$ to it. Thus

$$2u_2(t,r) + r^2 G(t,r) = 0 \tag{15}$$

This condition fixes $u_2(t,r)$. Next highest derivative occurs in $\tilde{V}_P$ thus we nullify h_0 and h_1 by the conditions

$$u_0(t,r) + \partial_t u_2(t,r) + h_0(t,r) = 0, \tag{16}$$

and

$$u_1(t,r) + \partial_r u_2(t,r) - \frac{u_2(t,r)}{r} + h_1(t,r) = 0. \tag{17}$$

Remarkably we do not have to solve differential equation to find gauge functions u_i. The only derivatives we have to evaluate is that of $-G(t,r)(= 2u_2)$, which is assumed to be known.

Once we find gauge functions u_i, we can compute the rest components of metric perturbation in Regge-Wheeler gauge. Finn's slow motion formalism [3] to evaluate quasinormal damping time (or equivalently energy loss rate) utilizes coefficient of $\overset{2}{T}_{lm}$ in Regge-Wheeler gauge, thus $r^2 K(t,r) + 2re^{-2\lambda} u_1$ in our notation.

References

[1] Nollert H-P., Class. Quantum Grav. 16, R159 (1999)

[2] Regge T. and Wheeler J.A., Phys. Rev. 108, 1063 (1957)

[3] Finn L.S., MNRAS, 222, 393 (1986)