Dummy's note: (3) Vorticity conservation in perfect fluid

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1 Conservation of vorticity 2-form in Newtonian hydrodynamics

We derive the equation of vorticity conservation for barotropic fluid. Euler equation is written as [1]

$$(\partial_t + \pounds_{\bar{v}})\tilde{v} = -\tilde{d}\left(h + \Phi - \frac{1}{2}v \cdot v\right), \tag{1}$$

where $h := \int(1/\rho)dp$.

Taking the exterior derivative of both hand sides we obtain $\tilde{d}(\partial_t + \pounds_{\bar{v}})\tilde{v} = 0$. Notice that $\underline{\tilde{d} \text{ and } \partial_t}$ and $\underline{\tilde{d} \text{ and } \pounds_{\bar{v}}}$ are commutable, we obtain

$$(\partial_t + \pounds_{\bar{v}})\tilde{d}\tilde{v} = 0. \tag{2}$$

Here $\tilde{d}\tilde{v}$ is a vorticity 2-form. This shows that vorticity 2-form is conserved along the fluid motion.

1.1 ex.: Incompressible 2-dim fluid on a surface of a rotating sphere

Suppose we have a fluid on a 2-sphere whose coordinate is $dl^2 = d\theta^2 + \sin^2\theta d\varphi^2$. We assume that the veloticy is tangential to the surface

$$\tilde{v} = v_\theta \tilde{d}\theta + v_\varphi \tilde{d}\varphi, \quad \bar{v} = v^\theta \partial_\theta + v^\varphi \partial_\varphi. \tag{3}$$

From an identity of Lie derivative of p-form [2], we have

$$\pounds_{\bar{v}}\tilde{d}\tilde{v} = \tilde{d}\left(\tilde{d}\tilde{v}(\bar{v})\right) + \tilde{d}\tilde{d}\tilde{v}(\bar{v}), \tag{5}$$

where the second term vanishes. While

$$\tilde{d}\tilde{v} = (\partial_\theta v_\varphi - \partial_\varphi v_\theta)\tilde{d}\theta \wedge \tilde{d}\varphi, \tag{6}$$

we have

$$\tilde{d}\tilde{v}(\bar{v}) = v^\theta(\partial_\theta v_\varphi - \partial_\varphi v_\theta)\tilde{d}\varphi - v^\varphi(\partial_\theta v_\varphi - \partial_\varphi v_\theta)\tilde{d}\theta. \tag{7}$$

Therefore

$$\tilde{d}(\tilde{d}\tilde{v}(\bar{v})) = \left(\partial_\theta(v^\theta\omega_r) + \partial_\varphi(v^\varphi\omega_r)\right) = \pounds_{\bar{v}}\tilde{d}\tilde{v}, \tag{8}$$

where we defined $\omega_r = \partial_\theta v_\varphi - \partial_\varphi v_\theta$. Whereas $\partial_t\tilde{d}\tilde{v} = \partial_t\omega_r\tilde{d}\theta \wedge \tilde{d}\varphi$, thus we have

$$\partial_t\omega_r + \partial_\theta(v^\theta v\omega_r) + \partial_\varphi(v^\varphi v\omega_r). \tag{9}$$

[1] We follow the notation adopted in Schutz (*Geometrical methods of mathematical physics*, Cambridge University Press (1980)). A vector is denoted as $\bar{v}$, a form is denoted as $\tilde{\omega}$. $\tilde{d}$ means the exterior derivative operator, while $\pounds_{\bar{u}}$ is Lie derivative along vector field $\bar{u}$.

[2] For any p-form $\tilde{\omega}$, we have

$$\pounds_{\bar{v}}\tilde{\omega} = \tilde{d}(\tilde{\omega}(\bar{v})) + \tilde{d}\tilde{\omega}(\bar{v}). \tag{4}$$

If we use the equation of continuity

$$\partial_\theta(v^\theta \sin\theta) + \partial_\varphi(v^\varphi \sin\theta) = 0, \tag{10}$$

the equation reduces to

$$\partial_t \omega_r + v^\theta \partial_\theta \omega_r + v^\varphi \partial_\varphi \omega_r = 0, \tag{11}$$

which is the conservation of scalar vorticity ω_r along the fluid motion.

We can easily derive Rossby wave by coupling this with the equation of continuity.

2 Relativistic treatment

To get the relativistic version, we start from energy-momentum tensor of perfect fluid

$$\begin{aligned} T^{\alpha\beta} &= (e+p)u^\alpha u^\beta + p g^{\alpha\beta} && (12)\\ &= \rho h u^\alpha u^\beta + p g^{\alpha\beta} && (13)\\ &= \pi^\alpha \rho u^\beta + p g^{\alpha\beta}, && (14) \end{aligned}$$

where e is energy density, ρ is rest mass density and h is enthalpy defined by $h := (e+\rho)/\rho$. Specific momentum π^α is defined as $\pi^\alpha := h u^\alpha$.

Conservation of energy momentum is thus written as,

$$\begin{aligned} 0 = \nabla_\beta T^\beta_\alpha &= \nabla_\beta \pi_\alpha \cdot \rho u^\beta + \pi_\alpha \nabla_\beta(\rho u^\beta) + \nabla_\alpha p && (15)\\ &= 2\nabla_{[\beta}\pi_{\alpha]}\rho u^\beta + \nabla_\alpha \pi_\beta \cdot \rho u^\beta + \pi_\alpha \nabla_\beta(\rho u^\beta) + \nabla_\alpha p. && (16) \end{aligned}$$

The 2nd and 4th term is combined to give,

$$\begin{aligned} &\nabla_\alpha \pi_\beta \cdot \rho u^\beta + \nabla_\alpha p && (17)\\ =\ &\rho \nabla_\alpha(\pi_\beta u^\beta) - \rho \pi_\beta \nabla_\alpha u^\beta + \nabla_\alpha p && (18)\\ =\ &-\rho \nabla_\alpha h + \nabla_\alpha p. && (19) \end{aligned}$$

Therefore the conservation equation is,

$$2\nabla_{[\beta}\pi_{\alpha]}\rho u^\beta + \pi_\alpha \nabla_\beta(\rho u^\beta) - \rho \nabla_\alpha h + \nabla_\alpha p = 0. \tag{20}$$

Notice that from the conservation of baryon number, the second term on the left hand side vanishes. This equation is written in abstract form as

$$\tilde{d}\tilde{\pi}(\bar{u}, \sqcup) - \tilde{d}h + \frac{1}{\rho}\tilde{d}p = 0, \tag{21}$$

where the parenthesis of the first term means that $\bar{u}$ is inserted to the first slot of 2-form. We have identity (see Eq.(4))

$$\tilde{d}(\tilde{d}\tilde{\pi}(\bar{u}, \sqcup)) = -\tilde{d}\tilde{d}\tilde{\pi}(\bar{u}) + \pounds_{\bar{u}}\tilde{d}\tilde{\pi}. \tag{22}$$

The first term on the right hand side vanishes. Using this and taking the exterior derivative of Eq.(20), we obtain

$$\pounds_{\bar{u}}\tilde{d}\tilde{\pi} - \frac{1}{\rho^2}\tilde{d}\rho \wedge \tilde{d}p = 0. \tag{23}$$

The first term shows change of vorticity 2-form $\tilde{d}\tilde{\pi}$ along fluid flow $\bar{u}$. Therefore this is the vorticity conservation in relativistic language. Note that the second term vanishes when the fluid is barotropic $p = p(\rho)$.